CS598: High-Order Methods for PDEs
Final Project — Due Thursday, May 12

1. Use a two-dimensional spectral element method to solve the incompressible Navier-Stokes equations with thermal transport,

$$\frac{\partial u_i}{\partial t} + \mathbf{u}\cdot\nabla u_i = -\nabla p + \frac{1}{Re}\nabla^2 u_i + f_i, \qquad i = 1,2 \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \frac{1}{Pe}\nabla^2 T, \tag{3}$$

in the 2D annulus having inner diameter $D_0 = 1$ and outer diameter $D_1 = 3$. The boundary conditions for this problem are $\mathbf{u} = (u_1, u_2) \equiv 0$ on $\partial\Omega$ and $T = 0$ on the outer surface and $T = 1$ on the inner surface. Take as an initial condition $\mathbf{u} = 0$ and $T = (R_1 - r)/(R_1 - R_0)$, where R_0 and R_1 are the respective inner and outer radii.

To match the figures on the left, we will take Grashof number $Gr = 120,000$ and Prandtl number $Pr = 0.8$. The Reynolds number is $Re = Gr^{\frac{1}{2}}$ and the Peclet number is $Pe = RePr$.

The forcing term is based on a Boussinesq approximation, which under the given nondimensional-

Figure 1: Left: computational domain with unit-diameter inner cylinder. Center: experimental streamlines at Gr=120,000. Right: experimental isotherms.

ization will be,

$$f_1 = 0 \tag{4}$$

$$f_2 = T. \tag{5}$$

$$\tag{6}$$

Note that f_2 will be large in regions where T is large, which matches our intuition that hot fluid will rise.

I would recommend a 5×1 array of spectral elements of order $N = 15$. I would also suggest that the mesh be oriented such that one of the element boundaries is at the $\theta = 90$ mark, that is, point upward from the inner cylinder to the outer. You can try more or less resolution, but this gives reasonable results in reasonable time. You might start with slightly lower N while debugging. It should run pretty fast.

Note that your *pressure system* will have a singularity associated with the mean. As long as the right-hand side of your pressure system is orthogonal to the "1"-vector, this should cause no difficulty for conjugate gradient iteration. (Be careful, however, if you precondition your pressure solve that the preconditioner also produces vectors with zero mean.)

To get to a settled steady state solution under these conditions will require integrating out to a final time of $t \approx 140$, but you will see the correct trends well before that (e.g., around $t = 5$–10).