CS598: High-Order Methods for PDEs — Assignment 1 — Due Thursday, Jan. 28

These exercises are intended to familiarize you with some tools used to derive, analyze, and understand timestepping schemes. Estimated write-up is about two to three pages, including figures.

1. Find (closed form) the roots of the following:

a) $x^2 + \epsilon x - 1 = 0,$

b) $\epsilon x^2 + x - 1 = 0,$

in $\lim \epsilon \longrightarrow 0$. Comment on the root structure. Which of these, (a) or (b), would be considered a singular perturbation?

2. It is proposed to use AB3 in conjunction with 2nd-order centered finite differences to solve $u_t + u_x = 0$ on the periodic domain $\Omega = [0, 1]$. The grid spacing is uniform: $x_j = j\Delta x$.

a) What is the largest allowable timestep from a stability standpoint?

b) What is the largest allowable step when the spatial discretization is replaced with 4th-order centered finite differences using a 5-point stencil on $[u_{j-2} \dots u_{j+2}]$?

c) As an alternative to a 5-point stencil, one can consider replacing u_t with

$$\frac{d}{dt}[\alpha u_{j-1} + \beta u_j + \alpha u_{j+1}]$$

for an optimal choice of constants α and β.

Use Taylor series expansions to find the (α, β) pair that will increase the order of spatial accuracy when u_x is based on the standard 2nd-order stencil. What is the leading-order truncation error (in space) ? How does it compare with the spatial error for case (b)?

d) Modify the code `adv_1d.m` to run this new scheme and demonstrate that your error estimates are correct, both with respect to the order of accuracy and the relative size of the constants.

e) What is the stability limit (maxium Δt or equivalently, $CFL := \Delta t U/\Delta x$) for this new scheme when using AB3 timestepping?

f) Is the leading-error order dispersive? Or diffusive? Are the dispersive errors trailing or leading waves?

HINTS for Problem 2: The idea here is that when discretizing $u_t = -cu_x$ in space one can modify both sides of the equation to increase the spatial order of accuracy.

Let's momentarily forget the time-derivative part and simply try to find f satisfying

$$f = -cu_x \tag{1}$$

using finite differences on $x_j = jh$, $j = 0, \ldots, n-1$ with periodic boundary conditions on [0,1]. One can of course use:

$$f_j = -\frac{c}{2h}(u_{j+1} - u_{j-1}) + O(h^2). \tag{2}$$

We could also use

$$\alpha f_{j-1} + \beta f_j + \alpha f_{j+1} = -\frac{c}{2h}(u_{j+1} - u_{j-1}) + O(h^4), \tag{3}$$

or, equivalently,

$$M\underline{f} = -C\underline{u} + O(h^4), \tag{4}$$

where M=tridiag(α, β, α) and C is our usual 2nd-order discretization for cu_x, both with periodic wrap.

The $O(h^4)$ truncation error will hold true only for a particular (α, β) pair. The basic approach to finding this pair is to expand f about x_j with a Taylor series and then use (1) to convert this into expansion for u so that you can eliminate terms.

Note, however, that when you *solve* (as opposed to derive) the system (4), you will have something like $\underline{f} = -M^{-1}C\underline{u}$. It turns out that this rational approximation gives an improved error constant (but no change in order, which remains 4) over simply moving the other terms in the Taylor series expansion to the rhs. I found that the fastest way to derive this improved error constant is to look at the associated eigenproblem for the time-dependent problem. That is, set $f := u_t$ (from where we started) and consider

$$\frac{d\underline{u}}{dt} = -M^{-1}C\underline{u}. \tag{5}$$

As always, we're interested in the eigenvalue distribution of $M^{-1}C$, i.e.,

$$-M^{-1}C\underline{\phi}_k = \lambda_k \underline{\phi}_k, \tag{6}$$

or equivalently,

$$-C\underline{\phi}_k = \lambda_k M \underline{\phi}_k. \tag{7}$$

For spatial error analysis, we wish to relate λ to the eigenvalues of the continuous problem, $\tilde{\lambda}_k = i2\pi k$. Fortunately, M and C share the same eigenvectors

$$\left(\underline{\phi}_k\right)_j = e^{i2\pi k x_j}, \tag{8}$$

and derivation of a closed-form expression for the discrete eigenvalues λ_k follows by applying the finite difference (or sum) operators C and M to $\underline{\phi}_k$.