CS598: High-Order Methods for PDEs
Assignment 4 — Due Tuesday, Feb. 23

These exercises are intended to familiarize you with some tools used to derive, analyze, and understand the Galerkin formulation in 1D. Estimated write-up is about two to three pages, including figures. (Note, semhat.m and dependencies are provided. Please let me know if you have difficulty finding python equivalents.)

1. Consider the ODE,

$$-\frac{\partial^2 u}{\partial x^2} = f(x), \qquad u(0) = u(1) = 0,$$

with $f(x) = \sin k\pi x$. The exact solution is $\tilde{u}(x) = (k\pi)^{-2} \sin k\pi x$.

Solve this ODE using a spectral Galerkin scheme with nodal bases on the GLL points $\xi_j \in \hat{\Omega} := [-1, 1]$. (You will have to translate your problem from $\Omega := [0, 1]$ to $\hat{\Omega}$.)

Demonstrate spectral convergence by plotting the maximum pointwise (normalized) error,

$$err := \frac{||u - \tilde{u}||_\infty}{||\tilde{u}||_\infty},$$

vs. N on a semilogy plot for $k = 1$ and $k = 20$. Comment on your results.

2a. Alter your code from the preceding exercise to solve

$$-\frac{\partial^2 u}{\partial x^2} = \sin \pi x, \qquad u(0) = 1, \qquad u'(1) = 0,$$

and plot error vs. N.

2b. Now consider

$$-\frac{\partial^2 u}{\partial x^2} = \sin \pi x, \qquad u(0) = 0, \qquad u'(1) = 1,$$

and plot error vs. N.

2c. Consider the Robin boundary condition case

$$-\frac{\partial^2 u}{\partial x^2} = \sin \pi x, \qquad u(0) = 0, \qquad \alpha u'(1) + \beta u(1) = \gamma,$$

and plot error vs. N for the case $\alpha = 5$, $\beta = 10$, $\gamma = 2$. What are the conditions on α, β, and γ to guarantee well-posedness (i.e., solvability) of this problem?

3. Let's reconsider the ODE of problem (1) on $\Omega = \hat{\Omega}$,

$$-\frac{\partial^2 u}{\partial x^2} = f(x), \qquad u(-1) = u(1) = 0,$$

but now with a solution having limited regularity. Specifically, we take $f(x) = |x|^k$. The exact solution is

$$\tilde{u}(x) \quad = \quad \frac{1 - |x|^{k+2}}{(k+1)(k+2)}.$$

We wish to confirm the theoretical convergence result

$$||u - \tilde{u}||_0 \;\; \leq \;\; C N^{-s} ||\tilde{u}||_s, \;\; s \geq 2$$

[*Quarteroni and Valli (6.2.19). See also wikipedia page on spectral element methods for a similar result.*] Here, $||\tilde{u}||_s$ is the Sobolev norm,

$$||\tilde{u}||_s \quad := \quad \left[\sum_{j=0}^{s} \int_{\hat{\Omega}} \left| \frac{d^j \tilde{u}}{dx^j} \right|^2 \; dx \right]^{\frac{1}{2}}$$

Plot the error for $N = 2, \ldots, 40$ for each case, $k = 0, \ldots, 11$. Prepare two graphs, one for k even, one for k odd. Discuss your findings and compare with the theory. Do your results change if you use exact integration?