CS598: High-Order Methods for PDEs
Assignment 9 — Due Tuesday, April 25

1. Use a two-dimensional spectral element method to solve Laplace's equation $-\nabla^2\psi = 0$ in the annular domains shown in Fig. 1. The inner circle is of diameter $D_i = 1$ and the outer has diameter $D_o = 3$. Take as boundary conditions $\psi = 0$ on the inner ring and $\psi = 1$ on the outer ring. You can verify the correctness of your code for the case on the left, for which ψ satisfies, in polar coordinates,

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi}{\partial r}\right) \;=\; 0, \tag{1}$$

The solution is $\psi = A\ln r + B$, where A and B are constants chosen so that ψ satisfies the boundary conditions. (As always, demonstrate that your code is behaving as expected.)

Note: It's probably sufficient to use CG for this problem and to precondition with the identity matrix, i.e., no preconditioning.

2. Solve the advection diffusion equation

$$\frac{\partial u}{\partial t} \;+\; \mathbf{c}\cdot\nabla u \;=\; \nu\nabla^2 u, \tag{2}$$

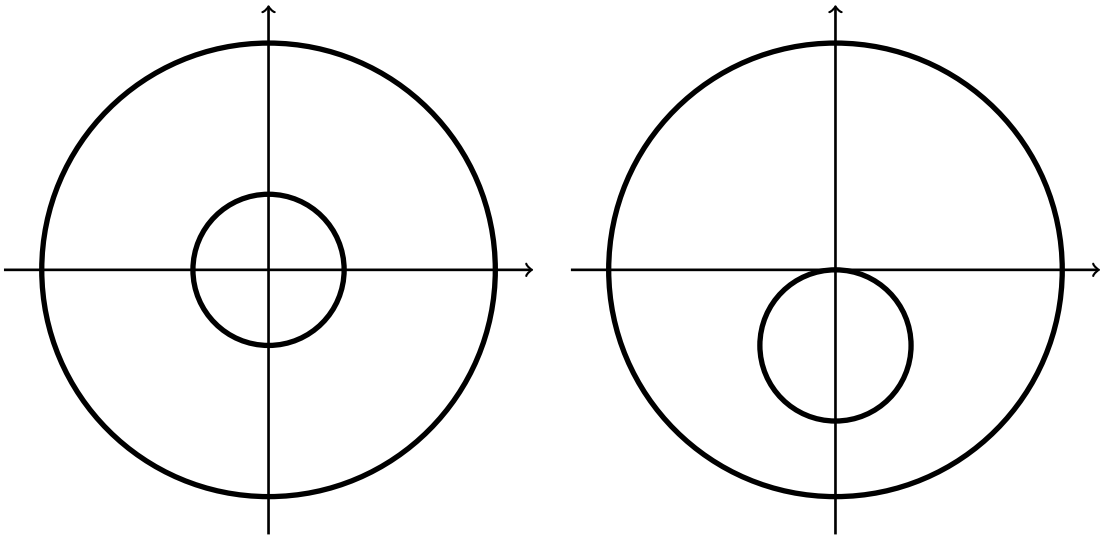


Figure 1: Computational domains.

with an initial Gaussian pulse of amplitude 1 and position and width such that its value is effectively 0 at the domain boundaries and $\nu = .01$. Take the convecting field to be $\mathbf{c} = (-\psi_y, \psi_x)$ from the solution of Problem 1 and use boundary conditions of your choosing for u on $\partial\Omega$.

You can verify your code for the case on the left by the following tests.

- Set ν=0 and $\mathbf{c} = (-y, x)$, which is the case of plane rotation. Your result should just move around and return to its initial condition, to within max(TE,RE), where TE:=truncation error and RE:=round-off error. Verify that TE scales as expected.
- Set ν=1 and $\mathbf{c} = (0, 0)$ and see that your solution evolves to the solution of Problem 1.

The first test checks your advection operator, the second checks your diffusion operator.

Note, you'll probably want to use CG to solve the system $H\underline{u} = \underline{b}$, where $H = \Delta t\nu A + B$. I would suggest to use the diagonal matrix B as the precontioner. Each iteration will involve matrix-vector products of the form $H\underline{p} = Q^T H_L Q\underline{p}$ where Q is the global-to-local map and Q^T is the local-to-global map (with summation!) that we discussed before and which is outlined in Chapter 4 of DFM02. To do this, you will need to establish a numbering scheme for your elements. I suggest to make the annular domain on the left by wrapping a periodic $E \times 1$ array of elements (say), as pictured in the next section. For example, if the starting rectangular array is $(r, s) \in [0, L_x] \times [0, L_y]$, set

$$\theta = \frac{2\pi r}{L_x} \tag{3}$$

$$\rho = 1 + 2\frac{L_y - s}{L_y} \tag{4}$$

$$x = \rho\cos\theta \tag{5}$$

$$y = \rho\sin\theta. \tag{6}$$

In principle, this mapping should preserve right-handedness. You can then generate the domain on the right by morphing the coordinates $(x, y) \longrightarrow (x', y')$ with a downward shift in y whose amplitude is -0.5 at the inner circle and 0.0 at the outer circle.

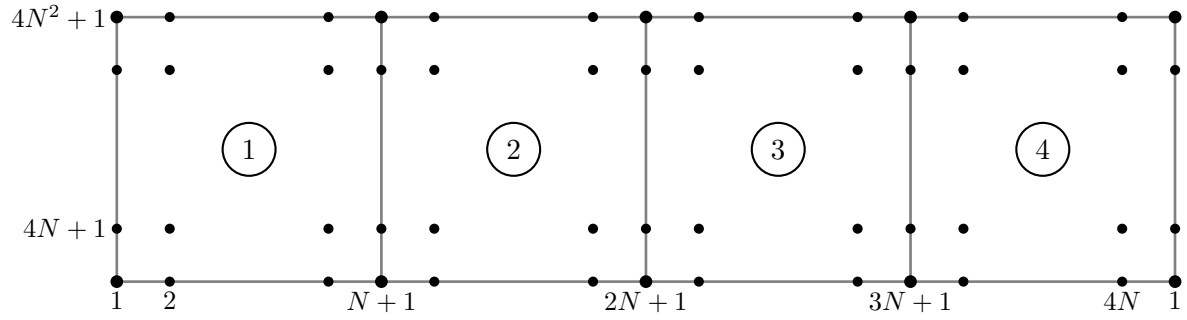
Please notify me if you have any questions.

Grid Point Numbering Example

Assembly of the degrees-of-freedom for FEM and SEM applications requires some (any, almost arbitrary) enumeration of the global degrees-of-freedom that maps to the local degrees-of-freedom. As mentioned in class, the numbering usually goes the other way around—for any element number $e \in \{1, \ldots, E\}$ and local index pair $(i, j) \in \{1, \ldots, N+1\}^2$ (or (i, j, k) in 3D), one associates a global index $g = g(i, j, e)$ that will be unique for any function in $X^N \subset \mathcal{H}^1$. That is,

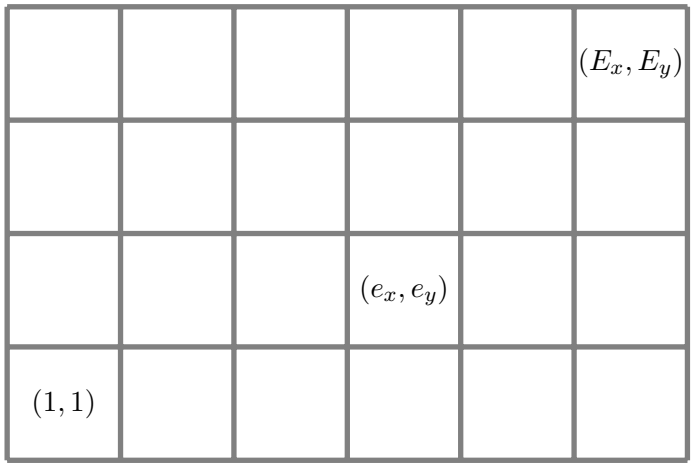
$$g(i, j, e) \;=\; g(i', j', e') \quad \text{if} \quad \mathbf{x}(i, j, e) = \mathbf{x}(i', j', e'). \tag{7}$$

(Functions in L^2 may be discontinuous at element boundaries, hence, they need not have a unique local/global representation.) An example of a global vertex-numbering scheme is shown below for a 4×1 array of spectral elements that is periodic in the x-direction. Notice that, in addition to (7), periodicity imposes an additional topological constraint on the numbering.



We can generalize the above numbering scheme, which is designed to be periodic in the nominal "x" direction (i.e., the "r" direction in the pictured canonical reference frame), in the following way. Assume we have a lexicographically-ordered two-dimensional array of spectral elements, (e_x, e_y), e_x=1, …, E_x and e_y=1, …, E_y. We assign a global element number to each element as

$$e \;=\; e_x + (e_y - 1) \cdot E_x \;\in\; \{1, \ldots, E\} \tag{8}$$



where $E := E_x\, E_y$ is the total number of elements in the array. We also (arbitrarily) number each unique node in a global tensor-product array. Assume that $(\hat{\imath}, \hat{\jmath}) \in \{1, \ldots, \hat{m}\} \times \{1, \ldots, \hat{n}\}$ enumerates the array of vertices. Here, $\hat{m} = NE_x$ if the domain is periodic in x or $\hat{m} = NE_x + 1$ if the domain is not periodic in x. Then, for any spectral element $e = e_x + E_x(e_y - 1)$ and *local* index pair $(i, j) \in \{1, \ldots, N+1\}^2$, the associated *global* index g is

$$g = \hat{\imath} + \hat{m}\,(\hat{\jmath} - 1) \tag{9}$$

$$\hat{\imath} = (e_x - 1) \cdot N + i \tag{10}$$

$$\hat{\jmath} = (e_y - 1) \cdot N + j. \tag{11}$$

Note that if the domain is periodic in x we must set $\hat{\imath} = \mod(\hat{\imath} - 1, \hat{m}) + 1$ so that the last value of $\hat{\imath}$ does not exceed $\hat{m}$.