## TAM 470 Introduction to Computational Mechanics: Assignment 3

## Due: Wednesday, October 12, 2016.

**1.** Use second-order centered finite differences to solve the eigenvalue problem on $[-1, 1]$,

$$-\frac{d^2u}{dx^2} \;=\; \lambda g(x)\, u(x), \;\; u(-1) \;=\; u(1) \;= 0,$$

where $g(x)$ is a given weight function. We are interested in the *smallest* eigenvalue, $\lambda := \min_k \lambda_k$.

**1a. [10 points]** Demonstrate that you have 2nd-order accuracy by comparing your estimated $\lambda$ to the exact eigenvalue, $\tilde{\lambda}$, for the case $g(x) = 1$.

**1b. [15 points]** Now find $\lambda$ for $g(x) = 1 - x^2$ with $h = 2^{-k}$, $k = 1, 2, 3, 4, 5$. Use two rounds of Richardson extrapolation to improve the answer. Estimate the error for this improved result.

**2.** Consider the PDE,

$$\frac{\partial u}{\partial t} \;=\; \frac{\partial^2 u}{\partial x^2}, \qquad u(x, t = 0) = u^0(t), \qquad u(0, t) = u(1, t) = 0. \tag{1}$$

Solve this PDE using 2nd-order centered differences in space and trapezoidal rule in time.

**2a. [15 points]** Demonstrate that you have 2nd-order accuracy in space (hold $\Delta t$ fixed and vary $\Delta x$) and time (hold $\Delta x$ fixed and vary $\Delta t$) for the initial condition $u^0 = \sin \pi x$.

**2b. [10 points]** Now consider the initial condition $u^0 \equiv 1$, with discretization parameters $\Delta t = 0.02$ and $\Delta x = 0.02$. What happens?

**3.** Repeat the preceding exercise using BDF2 in time.
(Note: use BDF1 on the first time-step.)

**3a. [15 points]** Demonstrate that you have 2nd-order accuracy in space and time for the initial condition $u^0 = \sin \pi x$.

**3b. [10 points]** Consider the initial condition $u^0 \equiv 1$, with discretization parameters $\Delta t = 0.02$ and $\Delta x = 0.02$. What happens?

Give some figures or tables demonstrating that your codes are working as expected. Then, with codes verified, draw some conclusions about these timesteppers.

**4. [10 points]** Assuming $\lambda \in \mathbb{R}$, what is the asymptotic value of the growth factor $G$ for the trapezoidal rule as $\lambda \Delta t \longrightarrow -\infty$? (Show a derivation rather than just writing the answer.)

**5.** Consider the system describing a particle orbiting about the origin:

$$\frac{d}{dt} \left( \begin{array}{c} x \\ y \end{array} \right) = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \left( \begin{array}{c} x \\ y \end{array} \right),$$

with initial condition $\mathbf{x} := (x,y)^T = (1,0)^T$.

**5a.[15 points]** Use Euler-Forward to solve this system for a single period of the orbit, $T$. (What is $T$?) Make a table with $\Delta t = .1 \times 2^{-k}$, $k = 1, \ldots, 7$ that shows your final position $[x(T), y(T)]$ and the associated error. What is the order of accuracy for this scheme?

**5b.[10 points]** Without re-running the code, apply two rounds of Richardson extrapolation to the $[x(T), y(T)]$ values in your table. What is your final error in this case?

**6a.[10 points]** Repeat 5a using 3rd-order Adams-Bashforth. (Use AB1 and AB2 on steps 1 and 2.) What is your order of convergence? (Think/look carefully—report what you observe.)

**6b.[10 points]** What is the maximum allowable $\Delta t$ for stability in this case? (Look at the **AB3 Hints** listed below.) Show, on a single graph, plots of the solution $[x(t^n), y(t^n)]$, for $t^n = n\Delta t$, $n = 0, 1, \ldots$, with $\Delta t = \Delta t_c - .05$ and $\Delta t = \Delta t_c + .05$, with $\Delta t_c$ being the maximum value of $\Delta t$ for which the solution is stable. Both conditions are inaccurate. Which is more dangerous? Why?

**6c.[5 points]** (No coding required.) Suppose AB3 is applied to $\underline{u}_t = L\underline{u} + \text{IC}$, with

$$L = \left[ \begin{array}{cc} -1 & 1 \\ 0 & -20 \end{array} \right].$$

What is the maximum allowable $\Delta t$ for stability in this case?

**Adams-Bashforth Hints.**

- The first three AB$k$ formulations for $u_t = f(u,t)$ with uniform timestep size $\Delta t$ can be written in the following compact form:

$$u^n \;=\; u^{n-1} \;+\; \Delta t \left[\alpha_1 f^{n-1} \;+\; \alpha_2 f^{n-2} \;+\; \alpha_3 f^{n-3}\right],$$

| | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
|---|---|---|---|
| AB1: | 1 | 0 | 0 |
| AB2: | $\frac{3}{2}$ | $-\frac{1}{2}$ | 0 |
| AB3: | $\frac{23}{12}$ | $-\frac{16}{12}$ | $\frac{5}{12}$ |

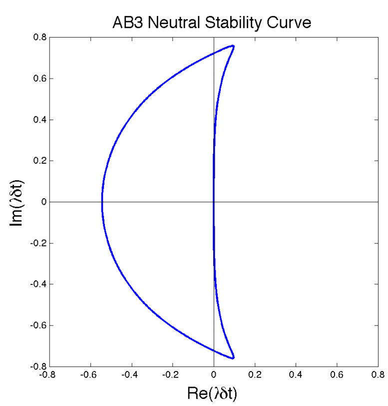
- A convenient way to implement this in matlab would be:

```
for k=1:nsteps; time=time+dt;
    if k==1; a1=1;     a2=0;      a3=0;    end;
    if k==2; a1=1.5;   a2=-.5;    a3=0;    end;
    if k==3; a1=23/12; a2=-16/12; a3=5/12; end;
    f = L*u; f3=f2; f2=f1; f1=f;
    u = u + dt*(a1*f1+a2*f2+a3*f3);
end;
```

- In the $\lambda\Delta t$-plane, AB3 has a neutral stability curve given by

$$\lambda\Delta t \;=\; \frac{1-e^{-i\theta}}{\frac{23}{12}e^{-i\theta} - \frac{16}{12}e^{-i2\theta} + \frac{5}{12}e^{-i3\theta}}, \qquad \theta \in [0, 2\pi], \tag{2}$$

which is shown in the figure below along with the matlab code used to generate the figure. The stability curve crosses the real axis at 0 and $\approx$ -.5454, and crosses the imaginary axis at $\approx \pm.7236i$.




```
i=sqrt(-1.); th=0:1000; ith=i*2*pi*th'/1000;
em1=exp(-ith); em2=em1.*em1; em3=em2.*em1; e0=1+0*em1;
a1 = 23/12; a2=-16/12; a3=5/12;
ab3 = (e0-em1)./(a1*em1+a2*em2+a3*em3);

plot(ab3,'b-','linewidth',2); hold on;
plot([-.8 .8],[0 0],'k-',[0 0],[-.8 .8],'k-');
axis square; axis tight;
title('AB3 Neutral Stability Curve','FontSize',18);
xlabel('Re(\lambda\deltat)','FontSize',18);
ylabel('Im(\lambda\deltat)','FontSize',18);
print -dpng ab3n.png
```

**7. 4-Credit Hour Option.** Stokes second problem concerns the development of boundary layers adjacent to an oscillating flat plate. The governing equation is

$$\frac{\partial \tilde{u}}{\partial t} = \nu \frac{\partial^2 \tilde{u}}{\partial y^2}, \qquad \tilde{u}(y=0,t) = \cos \omega t, \qquad \tilde{u}(y=\infty,t) = 0, \qquad \tilde{u}(y,t=0) = 0, \tag{3}$$

where $y$ is the direction perpendicular to the plate and $\tilde{u}$ is the velocity component parallel to the plate. At sufficiently long times, the boundary layer diffuses into the domain and the final (unsteady) solution is of the form

$$\tilde{u}(y,t) = e^{-y/d} \cos(\omega t - y/d), \tag{4}$$

where $d = \sqrt{2\nu/\omega}$. In the following questions, we will define our numerical error over a given period as

$$\epsilon := \max_n \max_{j=1}^{m-1} |\tilde{u}(y_j, t^n) - u_j^n|, \tag{5}$$

where $t^n$ covers one full period of the oscillation and $y_j$ covers the domain $y_j \in [0, L]$.

**Take $\nu = 0.1$ and $\omega = 5$ in the following.**

**7a. [20 points]** Solve Stokes 2nd problem numerically using a second-order scheme in time and space on $(y,t) \in [L,T]$, where $L$, the length of your domain, and $T$, the overall integration time, are chosen suitably large that you approach the exact solution. Justify (a priori) your choice for $L$. Justify (a priori) your choice for $T$. (By *a priori*, I mean estimating the values from some basic properties of the diffusion equation...i.e., the rate at which things decay in space and time.)

**7b. [10 points]** For fine grid spacing $h$, generate a log-log plot of $\epsilon(h, \Delta t)$ vs. $\Delta t$. What is the observed temporal order of accuracy?

**7c. [10 points]** For small $\Delta t$, generate a log-log plot of $\epsilon(h, \Delta t)$ vs. $h$. What is the observed spatial order of accuracy?

**7d. [10 points]** At your finest mesh and timestep size, do either $L$ or $T$ influence your solution? (How do you demonstrate to the reader, and yourself, that they do or do not?)