TAM 470 Introduction to Computational Mechanics

Midterm 1 Review Questions.

- *The questions below are presented in no particular order, with no particular weight.*
- *These are not all in exam-question format.*
- *They are simply review questions covering some of the topics discussed in class.*
- *I may post additional questions later on to help review topics not presented in this list.*
- *Be prepared to present your answers in a neat, concise, but complete, way.*

1. A kth-order finite difference method is used to estimate the mth-derivative of $f(x)$. Estimate the smallest error achievable in terms of the working precision of the computation, ϵ_M, the order of the derivative, m, and the order of the method, k. That is, estimate

$$\text{error}_{\text{min}} = \min_h \left| \frac{d^m f}{dx^m} - \left[\frac{\delta^m f}{\delta x^m} \right] \right|,$$

where $[g]$ denotes finite-precision evaluation (to relative precision $\leq \epsilon_M$) of the argument g. You may assume that the maximum of $|f|$ and all its derivatives are order unity in magnitude (e.g., as if $f(x)$ were like $\cos(x)$ or some other well behaved function in the region of interest).

Think. There are two parts the total error. What are they?

We are looking for an anser of the form $\text{error}_{\text{min}} = O(\epsilon_M^\alpha)$.

You need to find α (and show the intermediate steps by which you do so).

2. Consider the tensor-product function $f \in \mathbb{P}_2(\mathbf{x})$[1]

$$f(x,y,z) = \sum_{k=0}^{2} \sum_{j=0}^{2} \sum_{i=0}^{2} f_{ijk}\, l_i(x)\, l_j(y)\, l_k(z),$$

with Lagrange polynomial bases, $l_i(\xi) \in \mathbb{P}^2(\xi)$, $l_i(\xi_j) = \delta_{ij}$, and nodes $\xi_0 = -1$, $\xi_1 = 0$, and $\xi_2 = 1$.

Suppose $f_{ijk} = 0$ for all (i,j,k) triplets, save that $f_{000} = 1$.

What is $f(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$?

What is $f(\frac{1}{4}, -1, -1)$?

What is $f(\frac{1}{4}, -1, 1)$?

What is $\min_{(x,y,z)\in\Omega} f(x,y,z)$, where $\Omega = [-1,1]^3$ is the unit cube?

[1] More properly, as noted in class, $\mathbb{Q}_2$ in the finite element literature.

3. Consider interpolation of $f(x)$ on the interval $x \in [0,1]$ by an nth-order polynomial, $p_n(x)$. Show that, in the absence of round-off error, $|f(x) - p_n(x)| < .01$ for all $x \in [0,1]$ when $f(x) = \cos(x)$, $n = 4$, and you have *any* set of distinct node points $x_j \in [0,1]$, $x_j \neq x_i$ for $i \neq j$.

Hint. Start by writing the error formula for interpolation.

4. Consider the second-order centered finite difference approximation,

$$\frac{\delta^2 f_j}{\delta x^2} = \frac{1}{h^2}\left(f_{j-1} - 2f_j - f_{j+1} \right).$$

Use Richardson extrapolation with mesh spacing h and $2h$ to extend this formula to $O(h^4)$.

How many points (function evaluations) are required for this formula?

What is the coefficient for u_j in your $O(h^4)$ formula?

5. The composite trapezoidal rule with n panels of width $h = 1/n$ is used to evaluate

$$I \ := \ \int_0^1 f(x)\,dx,$$

where α and β are just constants. The results are shown in the table below:

n	I_h
2	21.0876
4	18.2541
8	13.0032
16	11.7823
32	11.4902
64	11.4180
128	11.4000

- Estimate the error, $|I_h - I|$, for the n=128 case.
- Use Richardson extrapolation to combine the I_h results for n=64 and n=128 cases to produce an improved approximation to I.

6. N-point Gauss quadrature is used to evaluate

$$I \quad := \quad \int_0^1 e^{-x} \sin(20\pi x)\, dx.$$

- **[5 points]** Estimate the value of N required before this method starts to exhibit asympotic convergence.
- **[10 points]** Justify your estimate with a short explanation.

7. 3rd-order backward-difference/extrapolation timestepping is used to advance $\frac{du}{dt} = L\underline{u}$, with

(a) $L = \begin{bmatrix} 0 & -1 \\ 16 & 0 \end{bmatrix}$ **(b)** $L = \begin{bmatrix} 0 & -1 \\ 0 & -16 \end{bmatrix}$ **(c)** $L\underline{u} = \alpha \dfrac{\delta^2 \underline{u}}{\delta x^2}$, $\alpha > 0$.

The BDF3/EXT3 stability region is shown in Fig. 1.

- What is the maximum stable timestep size Δt if BDF3/EXT3 is used to advance **(a)**?
- What is the maximum stable timestep size Δt if BDF3/EXT3 is used to advance **(b)**?
- What is the maximum stable timestep size Δt if BDF3/EXT3 is used to advance **(c)**?
- Which of these systems corresponds to a decaying solution?

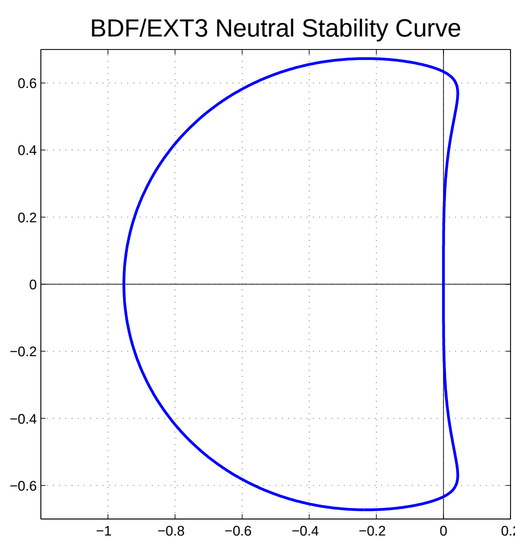

Figure 1: Stability region for BDF3/EXT3.

8. Data for a certain function is available at only two points, $f(x+h)$ and $f(x-2h)$, which is known to be smooth in this interval.

- With $f_1 := f(x+h)$ and $f_{-2} := f(x-2h)$, use Taylor series about x to derive an estimate of $f'(x)$.
- What is the leading-order error term? (Note, provide an expression for the coefficients in the leading order term, rather than just an $O(\cdot)$ expression.)

9. Consider $f(x) \in \mathbb{P}_1(x)$, with $f_j = f(x_j)$, $j = 0, 1$.

- **[10 points]** Write down the 2×2 derivative matrix D that generates $f'(x_j)$. That is,

$$\begin{pmatrix} f_0' \\ f_1' \end{pmatrix} = \begin{bmatrix} & D & \end{bmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

- **[10 points]** What are the units of D?

10. Consider the unsteady heat equation with $\alpha = 2.3 \approx \ln 10$.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = u^0(x).$$

with initial condition **(a)** $u^0(x) = \sin x$ and **(b)** $u^0(x) = \sin 2x$.

- Which of these decays fastest?
- What would be the ratio of the two solutions at time $t = 1.0$?