(Thermal) Mechanics:

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + q''' \quad + \begin{cases} \text{BC} \\ \text{IC} \end{cases}$$

- For k=constant (big assumption),

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial x^2} + \frac{q'''}{\rho C_p} \quad + \begin{cases} \text{BC} \\ \text{IC} \end{cases}$$

$$= \alpha \frac{\partial^2 T}{\partial x^2} + \tilde{q}'''$$

- Consider $q''' = 0$, let $u := T$, $u(0) = u(1) = 0$ (or, more generally, $u(L) = 0$).

Separation of Variables:

- Neglect I.C. for now.
- Just suppose

$$u(x,t) \;=\; c(t)\, s(x).$$

- Consider $s(x) \;=\; \sin(k\pi x)$.
- Satisfies BCs: $u(0) = u(1) = 0$.
- Insert into PDE:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{dc}{dt}\, s(x) \;=\; \alpha c(t) s''(x) = \alpha \frac{\partial^2 u}{\partial x^2} \\ &= -\alpha k^2 \pi^2\, c(t)\, s(x) \end{aligned}$$

- Time coefficient satisfies:

$$\begin{aligned} \frac{dc}{dt} &= -\alpha k^2 \pi^2 \, c(t) \\ \implies c(t) &= c^0 e^{-k^2\pi^2\alpha t}. \end{aligned}$$

- Denote this solution as

$$\begin{aligned} \hat{u}_k(x,t) &= c_k(t)\, s_k(x) \\ &= c_k(t)\, \sin(k\pi x) \\ &= c_k^0\, e^{-\alpha k^2 \pi^2 t}\, \sin(k\pi x). \end{aligned}$$

- **Main idea**: Choose c_k^0's to satisfy initial condition.

- **Aside:**
  - **Q:** What are the units of α?
  - **Q:** What are the units of k?
  - **Q:** What happens to the time-scale if domain is $x \in [0, \frac{1}{2}]$?

Behavior of Solutions for Various Wavenumber, k.:

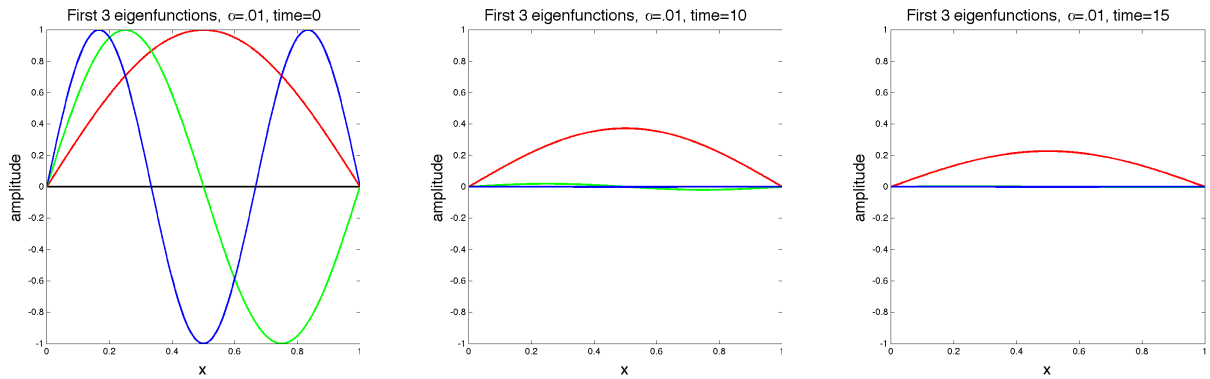


Figure 1: Solution behavior for first three eigenmodes, $c_k(t)\, s_k(x)$.

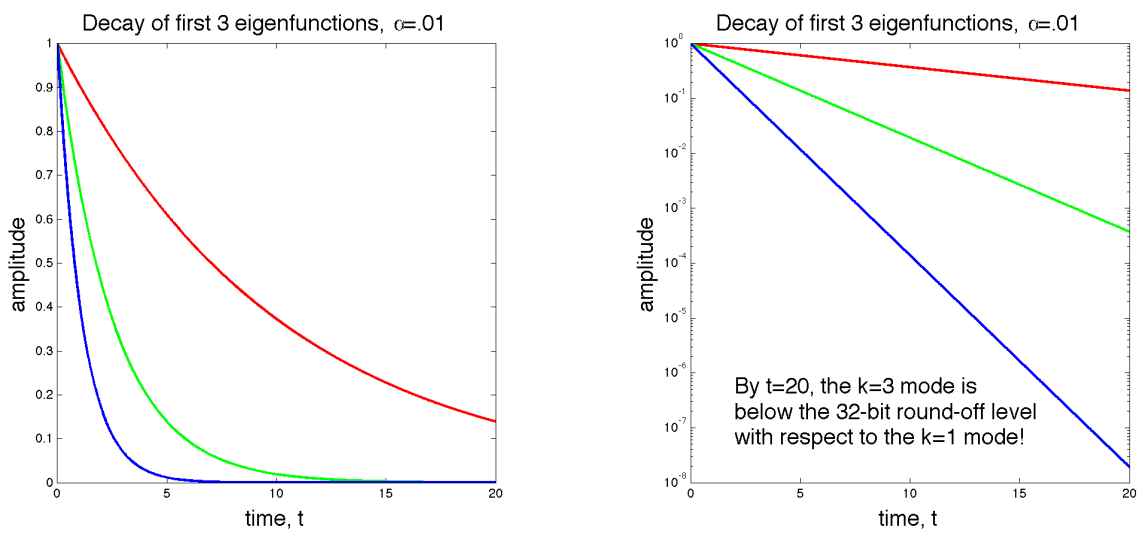


Figure 2: Coefficient amplitude for first three solutions, $c_k(t)\, s_k(x)$: (left) linear scale; (right) semi-log. Straight-line behavior in semi-log plot indicates exponential decay and reveals that by time $t = 20$ the 3rd mode has decayed significantly below even the 2nd mode. This observation is *not* evident in the linear-scale.

Constructing Full Solution:

- Choose $c_k^0 := c_k(0)$ values, $k = 1, \ldots,$ such that the initial condition is satisfied:

$$u(x, t = 0) = u^0(x).$$

- Problem is linear. Therefore, use *superposition*:

$$u(x,t) \;=\; \sum_{k=1}^{\infty} c_k(t) \sin(k\pi x).$$

- Satisfies $u_t \;=\; -\alpha u_{xx}$ and $u(0,t) = u(1,t) = 0$.

- To find c_k^0, use IC:

$$u^0(x) \;=\; \sum_{k=1}^{\infty} c_k^0 \cdot \sin(k\pi x).$$

- Sine functions are orthogonal on [0,1]:

$$(s_k,\, s_j) \;:=\; \int_0^1 \sin(k\pi x)\sin(j\pi x)\,dx \;=\; 0, \;\; k \neq j,$$

$$= \; \frac{1}{2}, \;\; k \;=\; j.$$

- This is easily shown by using integration-by-parts, twice.

$$(s_k,\, s_j) \;=\; \frac{k^2}{j^2}(s_k,\, s_j),$$

  which implies $(s_k,\, s_j) = 0$ for $i \neq j$.

- Now consider

$$\begin{aligned}
\int_0^1 \sin(k\pi x)\, u^0(x)\, dx &= \int_0^1 s_k(x) \left[\sum_{j=1}^{\infty} c_j^0\, s_j(x) \right] dx \\
&= \sum_{j=1}^{\infty} c_j^0 \cdot \int_0^1 s_k(x)\, s_j(x)\, dx \\
&= \sum_{j=1}^{\infty} c_j^0 \cdot (s_k,\, s_j) \\
&= \frac{1}{2}\, c_k^0.
\end{aligned}$$

- Therefore,

$$\begin{aligned}
c_k^0 &= 2 \int_0^1 \sin(k\pi x)\, u^0(x)\, dx \\
&= 2(s_k, u^0)
\end{aligned}$$

- The process of finding the c_k^0's is called *harmonic analysis*.
- It yields the *amplitude* of the frequency content associated with wavenumber k.

Finding the coefficients:

- It's annoying to have to integrate $\sin(k\pi x)u^0(x)$ for arbitrary u^0.

- Therefore, use *quadrature*. (yay!)

- We'll use trapezoidal rule (an excellent choice for *periodic* functions, but these functions are not generally periodic....)
- Define,

$$(f,g) \;:=\; \int_0^1 f(x)\,g(x)\,dx, \qquad \text{(continuous inner product)},$$

$$(f,g)_n \;:=\; h\sum_{j=0}^{n} f(x_j)\,g(x_j)w_j, \qquad \text{(discrete inner product)},$$

with quadrature weights $w_0 = w_n = \frac{1}{2},\ w_j = 1,\ j = 1,\ldots,n-1$.

- Orthogonality of basis functions, $s_k(x)$:

$$(s_k, s_j) \;=\; \frac{1}{2}\,\delta_{kl}$$

$$(s_k, s_j)_n \;=\; \frac{1}{2}\,\delta_{kl},\ \ 0 < k,\, l < n.$$

- Note that, because $s_k(0) = s_k(1) = 0$, we have

$$(s_k, f)_n \;=\; h\sum_{j=1}^{n-1} s_k(x_j)\,f(x_j)$$

$$= \;\frac{1}{n}\sum_{j=1}^{n-1} s_k(x_j)\,f(x_j).$$

Fourier (sine) Synthesis:

- Consider a *truncated* approximation (dropping superscript "0"):

$$
\begin{aligned}
u_n(x) &:= \sum_{k=1}^{n-1} s_k(x) c_k \;\approx\; u(x) \\
\underline{u}_n &:= \sum_{k=1}^{n-1} s_k(x_j) c_k, \;\; x_j \;=\; j \cdot h, \;\; h \;=\; \frac{1}{n}, \\
&:= S\underline{c}.
\end{aligned}
$$

- Think of the columns of S as the basis functions and $S\underline{c}$ as a linear combination of these basis functions.
- The matrix S is defined as

$$
S_{jk} \;:=\; s_k(x_j) \;=\; \sin(k\pi x_j) \;=\; \sin(\pi kj/n) \;=\; S_{kj}.
$$

- Note that

$$
\begin{aligned}
(S^T S)_{kj} &= \sum_{l=1}^{n-1} \sin(k\pi l/n) \sin(j\pi l/n) \\
&= n \cdot h \cdot \sum_{l=1}^{n-1} \sin(k\pi l/n) \sin(j\pi l/n) \\
&= n \cdot (s_k, s_j) \;=\; \frac{n}{2}\, \delta_{ij}.
\end{aligned}
$$

Thus, $S^T S = \dfrac{n}{2} I$.

- Use quadrature to find the basis coefficients, $\underline{c} = [c_1,\, c_2,\, \ldots c_{n-1}]^T$,

$$\begin{aligned} c_k &= 2(s_k, u)_n, \\ &= 2h \sum_{j=1}^{n-1} s_k(x_j)\, u_j, \\ \underline{c} &= 2hS^T \underline{u}_n. \end{aligned}$$

- Combining with the definition of $\underline{u}_n$, we have

$$\underline{c} \;=\; 2hS^T S\underline{c} \;=\; \underline{c}. \;\; \text{(no surprise)}$$

- We call $\underline{c} = 2hS^T\underline{u}_n$ the *discrete sine transform* (DST), and $\underline{u}_n = S\underline{c}$ *synthesis.*

- Generally, finding $\underline{c}$ is hard—except here, because $S^T = S$!

- We have

$$\underline{c} \;=\; 2h\, S^T \underline{u}_n \;=\; 2h\, S\underline{u}_n,$$

  which is just as easy as synthesis.

- Note that once we have $\underline{c}$, we can construct $u(x)$ on a *fine mesh*, just as we did in interpolation.

- Define $\underline{u}_m = \{u(\tilde{x}_i)\}$, $i = 0, \ldots, m$, $\tilde{x}_i = i/m$, and

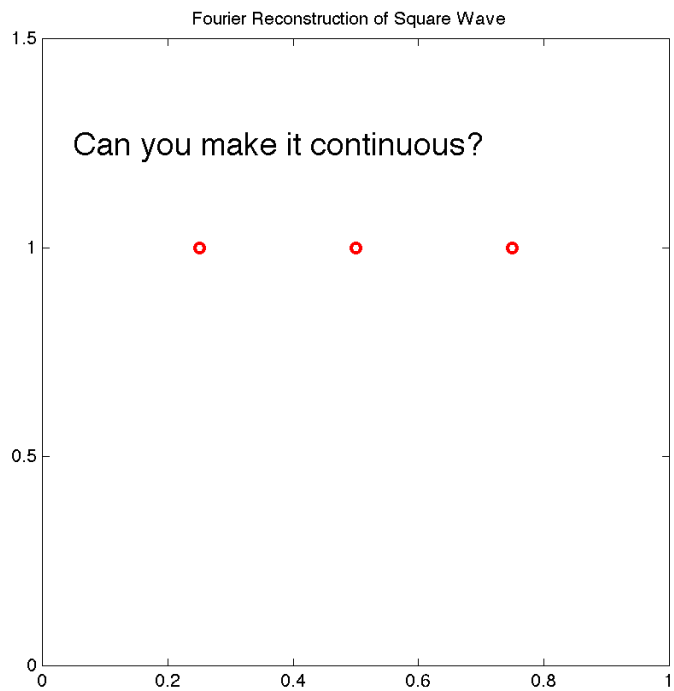
$$\begin{aligned} \underline{u}_m &= S_m \underline{c}, \\ (S_m)_{ik} &= \sin(k\pi\tilde{x}_i). \end{aligned}$$

- S_m is a rectangular $(m+1) \times (n-1)$ matrix with $m > n$.

Example:

- Suppose your boss asks you to do a sine-reconstrunction of a square wave.
- Your answer:

```
n=4;
x=1:(n-1); x=x'/n; n1=n-1; h=1./n;
S=zeros(n1,n1);
for k=1:n1; S(:,k)=sin(k*pi*x); end;
u0=0*x + 1;
c=(2*h)*S'*u0;
u=S*c;
plot(x,u,'ro','linewidth',2); axis([0 1 0 1.5]); axis square
title('Fourier Reconstruction of Square Wave')
```



- Boss is unimpressed.
- Wants a continuous line for unimaginative management.

```
n=4;
x=1:(n-1); x=x'/n; n1=n-1; h=1./n;
S=zeros(n1,n1);
for k=1:n1; S(:,k)=sin(k*pi*x); end;
u0=0*x + 1;
c=(2*h)*S'*u0;
u=S*c;
plot(x,u,'ro-','linewidth',2); axis([0 1 0 1.5]); axis square
title('Fourier Reconstruction of Square Wave')
```

- matlab demo: `bq2.m`