These exercises are intended to familiarize you with some tools used to derive, analyze, and understand the Galerkin formulation in 1D. Estimated write-up is about two to three pages, including figures.

1. Consider the steady-state advection-diffusion equation with homogeneous Dirichlet conditions,

\[-\nu \frac{\partial^2 u}{\partial x^2} + c \frac{\partial u}{\partial x} = f(x), \quad u(0) = u(1) = 0,\]

with \(c = 1\) and \(f = 1\). We are interested in cases where the diffusion coefficient \(\nu\) is small.

The exact solution for this problem is

\[\tilde{u}(x) = \frac{1}{c} \left( x - \frac{e^{-c/\nu} - e^{c(x-1)/\nu}}{e^{-c/\nu} - 1} \right)\]

Plot this solution for, say, \(\nu = 0.1, .01, .001, .0001\). What do you notice? How does the boundary layer thickness change with \(\nu\)? (Is it linear in \(\nu\) or something different?)

We now consider numerical solution of (1) using the SEM.

1. For \(\nu = 0.01\), produce a semilogy plot of error vs. \(N\) for the single element case, \(E=1\) and the two element case, \(E=2\). You should be able to get close to machine precision. Here, take the error to be the maximum-modulus pointwise difference between your solution and (2) evaluated at your nodal points.

2. For \(E = 1\) and \(N = 25\), plot the exact solution \(\tilde{u}(x)\) and your numerical solution vs \(x\) on the same graph for \(\nu = 0.1, .01, .001, .0001\).

3. Repeat 2, but with \(N = 26\). Can you give a linear-algebraic interpretation of what you observe in 2 and 3?

4. For \(E = 2, N=20\), and \(\nu = .001\), consider solution of (1) with variable element lengths. Specifically, let \(|\Omega^2| = \beta\) and \(|\Omega^1| = 1 - \beta\).

Approximately what value of \(\beta\) minimizes the error? (My approach: plot the maximum pointwise error as a function of \(\beta\) on a log-log graph and read off the value of \(\beta\) where this function takes on its minimum.)

In 15 words or less, what is it that you are doing when you change \(\beta\)?

5. A stretching exercise. Look at the paper by Saul Teukolsky, Short note on the mass matrix for Gauss-Lobatto grid points, that is posted on the CS598 Docs page.

Consider the case for a single element in 1D with periodic boundary conditions. As always, we have \(B = Q^T B_L Q\), where \(B_L\) is \((N+1) \times (N+1)\) and \(B\) is \(N \times N\). Consider the full mass matrix \((\tilde{B}_L)_{ij} := \int h_i(\xi) h_j(\xi) d\xi\) and the usual diagonal mass matrix \(B_L = \text{diag}(\rho_j)\). Will Teukolsky’s ideas extend to \(\tilde{B}\) and \(B\)? (I do not know the answer to this question... I have a hunch, but that’s all at the moment.)